The intersection product as the chromatic polynomial (after Aluffi)

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Main reference

Aluffi, P.: A blow-up construction and graph coloring, Discrete Mathematics **145**, 1995, 11-35.

Basic problem/philosophy

Given a *simple* graph G, attach a variety to it V_G such that one can translate combinatorial data on G into algebro-geometric data on V_G. More precisely:

Goal Create an intersection-theoretic interpretation of the proper coloring problem for a simple graph. Give geometric proofs of standard results in combinatorics.

A blowup construction

G: connected simple graph (ie, no loops or parallel edges with one conn comp). **Fix** some dimension n for the ambient proj. space.

- Place each vertex of G at some point in $\mathbb{P}^n$ s.t. all of these points are lin. indep.

- For each edge in G, draw a line joining the corresponding vertices in $\mathbb{P}^n$

$$\rightsquigarrow \text{ a curve } C \subset \mathbb{P}^n$$

- H: general hyperplane in $\mathbb{P}^n$. $C \cap H$ gives a config of points $\{e_k\}$, $k \in$ some finite index set. Each e_k corresponds to an edge of G (by contruction).

- x_r^d's: d-dim'l subspaces of $\mathbb{P}^n$ spanned by the e_k's. Order them by d such that x_k^0 is e_k's.

- y_r^d's: subspaces of $\mathbb{P}^n$ obtained by intersecting collections of x_r^d's **provided** these subspaces are not already in the list of x_r^d's.

Rk: y_r^d's are closed w.r.t intersections.
Let $V_0 = \mathbb{P}^n$ and

$$V_{d+1} \longrightarrow V_0$$

a blowup for $d \geq 0$ along the proper transform of x_r^d's and y_r^d's.

Blowups along dim d subspaces seperates the proper transforms of dim $d+1$ subspaces containing them
$\Rightarrow$ center of the blowups for each d are necessarily disjoint
$\Rightarrow$ a tower of varieties

$$V_G \xrightarrow{\pi_k} \dots \xrightarrow{\pi_1} V_0 = \mathbb{P}^n$$

π_i's are blowups.
Rk 1: k is finite since G is finite.
Rk 2: V_G depends on the dim n of V_0.
Call V_G the *graph variety* associated to the graph G.

Divisor classes in the graph variety

Notations

- H_0: pullback of the hyperplane class from $\mathbb{P}^n$.
- E_r^d: pullback of the exceptional divisors arising from the blowups of x_r^d.
- F_r^d: pullback of the exceptional divisors arising from the blowups of y_r^d.
- H_r^d: proper transforms of general hyperplanes containing x_r^d.
- R: dim of the subspace spanned by all the e_k's (i.e. x_k^0's.)

Rk: $R + 1 = \#$ of vertices of G.

Def. of $S(t)$, a polynomial in Pic $V_G[t]$:

$$S(t) := t^{R+1} H_0 - \sum_{d,r} t^{R-d} E_r^d$$

Recall (for a nonsingular scheme X):

Pic $X \stackrel{\text{def}}{=}$ ab. group of divisors of X mod lin. equivalence.

By construction,

H_0, the H_r^d and the F_r^d are a basis of Pic V_G.

$A_1(X) \stackrel{\text{def}}{=}$ ab. group of 1-cycles on X mod rat. equivalence.

Recall that a cycle

$$Z \sim_{\text{rat}} Z' \iff Z - Z' \sim_{\text{rat}} 0$$

and that for Z arbit. cycle of codim i, $Z \sim_{\text{rat}} 0 \iff \exists (Y_\alpha, f_\alpha)$ where Y_α is a closed irreducible subscheme of codim $i-1$ and f_α rat. fun on Y_α such that $\sum_\alpha \text{div}(f_\alpha) = Z$.

Poincare duality and intersection pairing

The pairing

$$A_1(X) \times \text{Pic } X \longrightarrow k$$

is given by

$$c_1(L) \cdot [Z] = \sum_i m_i [x_i] \rightsquigarrow \sum_i m_i$$

where

$$L \in \text{Pic } X \text{ and } [Z] \in A_1(X)$$

$m_i \in \mathbb{Z}$ and $[x_i]$ a 0-dim'l cycle.

Main theorem: I

By the pairing above,

$$\exists \gamma \in A_1(V_G) \text{ s.t. } \gamma$$

is dual to H_0w.r.t. the basis of Pic V_G

$$\iff H_0 \cdot \gamma = 1, H_r^d \cdot \gamma = 0, F_r^d \cdot \gamma = 0 \forall d, r$$

m: # of colors in the palette.

Main theorem:

# of ways G can be colored properly is

$$mS(m) \cdot \gamma$$

Corollary:

G can be colored properly with m colors iff

$$S(m) \cdot \gamma \neq 0$$

Example

G: complete graph with 4 vertices

$\rightsquigarrow$ 6 edges $\{x_k^0\}$, $k = 1, \ldots, 6$.

Know trivially that # of ways G can be colored properly with m colors is

$$m(m-1)(m-2)(m-3),$$

ie, all four vertices have different colors.

Want to show this fact using main theorem.

$$E_k^0 \cdot \gamma = 1, \quad k = 1, \ldots, 6,$$

$$E_r^1 \cdot \gamma = -2, \quad r = 1, \ldots, 4,$$

$$E_r^1 \cdot \gamma = -1, \quad r = 5, 6, 7,$$

$$E^2 \cdot \gamma = 6 \quad \text{(for the whole plane).}$$

So, $mS(m) \cdot \gamma$ equals:

$$m(m^3 - (6 \cdot 1)m^2 - (-2 \cdot 4 - 1 \cdot 3)m - (1 \cdot 6))$$

$$= \quad m(m-1)(m-2)(m-3)$$

as desired!

Compatibility with contractions

$x_r^d \rightsquigarrow$ collection of edges of graph G.

G_r^d: graph obtained by collapsing the collection corresponding to x_r^d

γ_r^d: dual of H_r^d in the basis $\{H_0, H_r^d, F_r^d\}$ of Pic $V_{G_r^d}$.

Corollary:

$mS(m) \cdot \gamma_r^d$ is the # of proper m-colorings of the contraction G_r^d.

Lattice of subspaces

$(L, \leq)$ a poset. L is called a **lattice** if $\forall\, x, y \in L$, the set $\{x, y\}$ has a least upper bound (join) in L *and* a greatest lower bound (meet) in L. Denote lub as $x \vee y$ and glb as $x \wedge y$.

$\mathcal{L} = \mathcal{L}(C)$: lattice of subspaces spanned by any (finite) collection C of points in $\mathbb{P}^n$.

$\mathcal{L}$ is partially ordered by inclusion of subspaces.

$\leq$: order by inclusion

$\vee, \wedge$: join and meet (resp.) of the lattice

x, y, z: elements of $\mathcal{L}$

$r(x) \stackrel{\text{def}}{=} 1 + k$ where $k = \text{dim}(x)$, when viewed as a subspace in $\mathbb{P}^n$.

0: empty set (minimum) of $\mathcal{L}$.

1: maximal subspace (spanned by all points)

$$r(0) \stackrel{\text{def}}{=} 0, \quad r(\mathcal{L}) \stackrel{\text{def}}{=} r(1).$$

Construction of the variety $V_{\mathcal{L}}$

- Choose $\mathcal{L}$ a family of subspaces of $\mathbb{P}^n$.

  $\mathcal{M}$: family of subspaces $\notin \mathcal{L}$ (obtained by intersection collections of elts of $\mathcal{L}$).

- Extend rank and $\leq$ to elts of $\mathcal{M}$.

- $V_{\mathcal{L}}$ is obtained from $\mathbb{P}^n$ by blowing up the proper transforms of nonzero x in $\mathcal{L}$ and $\mathcal{M}$ in order of inc. dim.

  $$V_{\mathcal{L}} \longrightarrow V^{(1)} \longrightarrow V^{(2)} \longrightarrow \ldots \longrightarrow V_0 = \mathbb{P}^n,$$

  where

  $$\dim V^{(2)} > \dim V^{(1)}, \quad \text{etc.}$$

For $G = \mathcal{L}$, the two constructions coincide.

H_x: class of proper transform of gen. hyperplane, $x \in \mathcal{L}$

E_x: pullback of the excep. divisor over $x \in \mathcal{L}$ $(x \neq 0)$

F_x: pullback of the excep. divisor over $x \in \mathcal{M}$.

$\forall\, x \in \mathcal{L}$, let γ_x be a 1-class such that $\gamma_x \cdot H_x = 1$, $\gamma_x \cdot H_y = 0 \quad \forall y \neq x$, and $\gamma_x \cdot F_z = 0 \quad \forall z \in \mathcal{M}$.

$$S(t) \stackrel{\text{def}}{=} t^{r(\mathcal{L})} H_0 - \sum_{x \in \mathcal{L}, x \neq 0} t^{r(\mathcal{L}) - r(x)} E_x.$$

Compatibility with deletion

C': subset of the set of rank 1 elements of $\mathcal{L}$

$\rightsquigarrow$ $\mathcal{L}(C')$: sublattice of $\mathcal{L}$ (referred to as the deletion of $\mathcal{L}$, analogous to deletion of edges in the case of graphs).

There exists a unique map $\rho : V_{\mathcal{L}} \longrightarrow V_{\mathcal{L}(C')}$.

Reason: Universal property of blowups!

$$\begin{array}{ccc} V_{\mathcal{L}} & \xrightarrow{\rho} & V_{\mathcal{L}(C')} \\ \downarrow & & \downarrow \\ V_0 & = & V_0 \end{array}$$

(Inverse image of ρ is a Cartier divisor in $V_{\mathcal{L}}$.)

Characteristic polynomial of $\mathcal{L}$

Möbius-Rota function on a given lattice $\mathcal{L}$:

$$\mu_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{Z}$$

is given by

$$\sum_{x \le y \le z} \mu_{\mathcal{L}}(x, y) = \begin{cases} 0 & \text{if } x \ne z \\ 1 & \text{if } x = z \end{cases} \quad \text{for } x \le z,$$

and

$$\mu_{\mathcal{L}}(x, z) = 0 \quad \text{for } x \not\le z.$$

Characteristic polynomial of $\mathcal{L}$ is:

$$P(\mathcal{L}, t) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{L}} \mu_{\mathcal{L}}(0, x) t^{r(\mathcal{L}) - r(x)}.$$

Main Theorem: II

Observation:

# of proper colorings of a graph G with m colors is

$$mP(\mathcal{L}, m),$$

where $\mathcal{L}$ is the lattice determined by G.

Theorem: For any lattice $\mathcal{L}$,

$$P(\mathcal{L}, t) = S(t) \cdot \gamma_0.$$

Sketch of the proof

From the definition of $\mu_{\mathcal{L}}$, observe that:

$$\mu_{\mathcal{L}}(0,0) = 1 = H_0 \cdot \gamma_0.$$

Need to show that

$$\mu_{\mathcal{L}}(0,z) = -E_z \cdot \gamma_0 \quad \text{for nonzero } z \in \mathcal{L}.$$

By definition of MR function, need to show that:

$$\mu_{\mathcal{L}}(0,0) + \sum_{0<y\leq z} (-E_y \cdot \gamma_0) = 0 \quad \text{for } z \neq 0.$$

By construction,

$$H_z = H_0 - \sum_{y\in\mathcal{L},0<y\leq z} E_y - \sum_{x\in\mathcal{M},x<z} F_x.$$

$\Longrightarrow$ LHS of $*$ is

$$\begin{aligned}(H_0 - \sum_{y\in\mathcal{L},0<y\leq z} E_y \quad &- \quad \sum_{x\in\mathcal{M},x<z} F_x) \cdot \gamma_0 \\ &= \quad H_z \cdot \gamma_0 = 0.\end{aligned}$$