Motivic Geometry of Feynman Amplitudes

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Outline

1. The Big (But Blurry!) Picture

2. Motives, descent and Grothendieck rings

3. Feynman integrals and number theory

4. Generating supermanifolds from realistic Feynman graphs

5. A universality result in the spirit of Belkale–Brosnan

6. Time permitting: Lie and Hopf algebras of Feynman graphs and of motives

The Big (But Blurry) Picture

1. Calculations of higher loop Feynman integrals after taking care of their divergences evaluate to single and multizeta values (MZV). It is an amazing fact that these numbers

$$\boxed{\zeta(s_1,\ldots,s_k) := \sum_{n_1>n_2>\ldots>n_k>0} \frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}, s_i \in \mathbb{C}, n_i \in \mathbb{N}}$$

   constructed by Euler and then refined by Kontsevich–Zagier have anything to do with physical values used for scattering calculations and such.

2. MZVs are interesting because they satisfy very interesting identities yet very little is known about them. Sample identities:

$$\begin{aligned} \zeta(2,1) &= \zeta(3) \text{ (Euler)}, \\ \zeta(a,b)+\zeta(b,a) &= \zeta(a)\zeta(b)-\zeta(a,b) \text{ for } a,b>1 \text{ (Borwein and/or Zagier)}. \end{aligned}$$

3. Since MVZs are involved, prevalent philosophy about the $\mathbb{Q}$-algebra MZV$[\frac{1}{2\pi i}]$ argues for a motivic interpretation (Deligne, Zagier, . . .), forcing us into the world of algebraic geometry

4. Combinatorics drives the motivic picture here. There are tons of numerical experiments due to Zagier, Broadhurst and others that show very combinatorial identities for MZVs. Dream: can we have a combinatorial theory of motives reflecting this combinatorics of MZVs driven by other well-understood combinatorial objects? The study of the moduli spaces $\overline{M_{0,n}}$ is a model.

5. Still– most work with scalar field theory. How about quantum field theories with fermions? These are my results with Marcolli.

6. Time permitting: Lie and Hopf algebra of Feynman graphs. A new Lie algebra of motives?

What I'll speak on is a very very small part of the story which I've had some involvement in. There are nice and authoritative surveys out already about this emerging subject as a whole!

Motives, descent and Grothendieck rings

The two main ideas driving Grothendieck's vision of a theory of motives:

1. "soften" the category of schemes to allow for an easier intersection theory and to "add" them– additive and abelian!

2. all cohomologies (Betti, étale, crystalline, . . .) factor through this category. NB: a cohomology theory is nothing but a contravariant functor from a suitable category of "nice" spaces to $\mathbf{Vect}_K$ satisfying a standard list of axioms like Poincare duality, Künneth formula, . . . (e.g: "Weil cohomology" for sm. proj. schemes).

Pure motives $=$ motives of smooth projective varieties,
Mixed motives $=$ motives of much more general and singular varieties.

While pure motives are understood, mixed motives are much more difficult! For one, they are a derived triangulated category as opposed to abelian! We are interested in something more intermediate to these two!

Mixed Tate motives

Roughly:

1. semisimple objects are pure motives (wrt homological equivalence),

2. $\mathrm{Ext}^i = 0$ for $i > 1$.

Over a base field k (let us take $k = \mathbb{Q}$ for technical simplicity), we define the following categories–

$$\begin{aligned} \mathbf{Var} &:= \text{varieties}, \\ \mathbf{Mot} &:= \text{pure motives wrt some adequate relation}, \\ \mathbf{MMot} &:= \text{mixed motives}, \\ \mathbf{MHodge} &:= \text{mixed Hodge structures}. \end{aligned}$$

Instead of working with these categories directly (which gets to be especially complicated with things like **MMot**) we will work with the associated Grothendieck ring

$$K_0(-)$$

of the appropriate category $(-)$. Various theories of descent (Bittner, Gillet–Soulé, . . .) assure us that we are not losing much by doing that.

The use of additive invariants

Consider (in a "good" category $\mathbf{C}$ of topological spaces) the topological Euler characteristic for a compact space X

$$\chi_{\mathrm{top}}(X) := \sum (-1)^k \mathrm{dim} H_c^k(X;\mathbb{Q}).$$

For compact $X, Y \in \mathrm{Obj}\ (\mathbf{C})$ with $Y \hookrightarrow X$ and $U = X \setminus Y$, we have

$$\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(Y) + \chi_{\mathrm{top}}(U).$$

Think of $K_0(\mathbf{C})$ as free group of homeomorphisms generated by the scissor relations

$$[X] = [U] + [Y].$$

Construct the category of fin. dim'l $\mathbb{Q}$-vector spaces $\mathbf{Vect}_{\mathbb{Q}}$ and define its Grothendieck ring– its the free group of object in $\mathbf{Vect}_{\mathbb{Q}}$ modulo the relation

$$V \sim U + W \iff 0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0 \text{ exact seq.}$$

The original definition of χ_{top} in terms of $K_0(\mathbf{C})$ and $K_0(\mathbf{Vect}_{\mathbb{Q}})$ can be thought of as a composition

$$\chi_{\mathrm{top}} : K_0(\mathbf{C}) \longrightarrow K_0(\mathbf{Vect}_{\mathbb{Q}}) \xrightarrow{\sim} \mathbb{Z}$$

with the last isom. being taking $\mathrm{dim}_{\mathbb{Q}} V$.

A de(s)cent result

Recall that a (pure effective Chow) motive is a pair (X, p) where X is smooth projective and p a projector in the ring of algebraic cycles in $X \times X$ modulo linear equivalence. The category of such objects $\mathbf{Mot}_{\mathrm{rat}}$ have a monoidal structure

$$(X_1, p_1) \oplus (X_2, p_2) = (X_1 \sqcup X_2, p_1 + p_2)$$

which means that by "abstract nonsense", can associate a Grothendieck ring $K_0(\mathbf{Mot}_{\mathrm{rat}})$ to it. One also in more general cases assigns

Tate twists $n \in \mathbb{Z}$

which comes from adding Tate objects $\mathbb{Q}(n)$ to the category of motives. (Cohomologically $\mathbb{Q}(1)$ is the formal inverse of $\mathbb{L} = H^2_{-}(\mathbb{P}^1)$.)

Theorem (Gillet–Soulé)**.** There exists an additive invariant

$$\chi_{\mathrm{mot}} : K_0(\mathbf{Var}) \longrightarrow K_0(\mathbf{MMot}).$$

For a smooth projective variety X, it assigns $\chi_{\mathrm{mot}}(X) = [(X, \mathrm{id}, 0)]$. To more general varieties it assigns an object $W(X)$ in the category of complexes over **MMot**.

Main thing to remember:

$$\begin{aligned} & \mathbb{L} &=& [\mathbb{A}^1] \in K_0(\mathbf{Var}_k) \\ \Longleftrightarrow & \mathbb{L} &=& \mathbb{Q}(-1) = [(\mathrm{Spec}\ k, \mathrm{id}, -1)] \in K_0(\mathbf{MMot}_k). \end{aligned}$$

$\mathbb{L}$ invertible in $K_0(\mathbf{MMot}_k)$ with inverse $\mathbb{Q}(1)$. Therefore there is a ring hom.

$$\chi_{\mathrm{mot}} : K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}] \longrightarrow K_0(\mathbf{MMot}_k).$$

Mixed Tate motives will be considered elements in the ring of Laurent polynomials

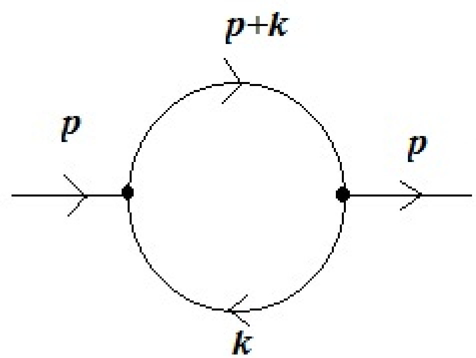
$$\boxed{\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathbf{MMot}_k).}$$

In the Hodge realization,

$$\mathbb{Q}(n) = (2\pi i)^n \mathbb{Q}.$$

Feynman integrals and number theory

Basic object in perturbative quantum field theory– graphs and loop integrals associated to the graphs.



$$\xrightarrow{(*)} \int \frac{1}{k^2} \frac{1}{(p+k)^2} dk.$$

$\boxed{(*) \text{ are the Feynman rules of the theory}}$

Physics: the (sum of) the renormalized integrals associated to graphs with increasing number of loops (= cycles) provides amplitudes for physical processes (scatterings, collisions, . . .).

and

Math: these graphs and the associated integrals systematically and in combinatorially organized families evaluate to multiple zeta values and is therefore (following the philosophy of Deligne, Goncharov and Manin in the study of $\overline{M_{0,n}}$) motivic.

From graphs to integrals Parametric form of a log-divergent Feynman integral associated to a graph with n edges and in D dimensions:

$$U(\Gamma) = \int_{\Sigma_n} \frac{\Omega}{\Psi_\Gamma^{D/2}}$$

where

$$\Omega = \sum_{i=0}^{n} (-1)^{i+1} t_i \, dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n$$

is the volume form in $\mathbb{P}^{n-1}(\mathbb{R})$, $\Sigma_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n_+ | \sum t_i \leq 1)\}$ is the topological simplex and the graph polynomial is defined as

$$\boxed{\Psi_\Gamma(t) := \sum_T \prod_{e \notin T} t_e}$$

where T is a spanning tree of Γ. Let l be the number of cycles ($=$ loops) of Γ. Log-divergence, stripped of its physical meaning, is the condition on the graph that

$$\boxed{n - \frac{D}{2} l = 0.}$$

The graph hypersurface

$$X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{P}^{n-1}$$

Basic object of study:

$$\boxed{(\mathbb{P}^{n-1} \setminus X_\Gamma, \Delta)}$$

where $\Delta = \{\prod_{i=1}^n t_i = 0\} \supset \partial\Sigma_n$ is the algebraic simplex.

Properties of X_Γ:

- Typically singular with singular locus of small codimension.
- Integral diverges whenever $X_\Gamma \cap \Delta \neq \emptyset$ – blow-ups needed!

We will focus on certain problems related to X_Γ and Ψ_Γ.

A main result needed:

$$\Psi_\Gamma(t) = \det M_\Gamma(t),$$

where $M_\Gamma(t)$ is constructed as follows.

- $n = \#E(\Gamma)$, $\ell = b_1(\Gamma)$ (# of loops), $\{l_1, \ldots, l_\ell\}$ basis of $H_1(\Gamma, \mathbb{Z})$

- $\eta_{ik} = \begin{cases} +1, & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation} \\ -1, & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation} \\ 0, & \text{otherwise} \end{cases}$

Then

$$(M_\Gamma)_{kr}(t) := \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}$$

for $t = (t_0, \ldots, t_{n-1}) \in \Sigma_n$, $t_n = 1 - \sum_{i=0}^{n-1} t_i$.

From general form to the parametric form

Let

- $p_i \in \mathbb{R}^D$: real variables associated to edges of Γ
- $s_k \in \mathbb{R}^D$: real variables associated to loops of Γ
- $q_i(p) := p_i^2 - m^2$: inverse propagator

Then it is a fundamental result in relativistic perturbative field theory that

$$\boxed{U(\Gamma) = \int \frac{1}{q_0 \cdots q_n} d^D s_1 \cdots d^D s_\ell = C_{\ell,n} \int_{\Sigma_n} \frac{dt_0 \cdots dt_{n-1}}{\det M_\Gamma(t)^{D/2}}}$$

where $C_{\ell,n}$ is a combinatorial factor depending on ℓ and n.

Taking stock... Let

$$\begin{array}{rl} P \to \mathbb{P}^{n-1} & \text{blowup along linear subspaces.} \\ Y_\Gamma & \text{strict transform of } X_\Gamma, \\ \widetilde{\Sigma} & \text{total inverse image of the simplex in } \mathbb{P}^{2n-1}. \end{array}$$

In terms of algebraic geometry, we can summarize all of with above with following problem- Is there a Hodge realization of the motive

$$\boxed{H^{2n-1}(P \setminus Y_\Gamma, \widetilde{\Sigma} \setminus (\widetilde{\Sigma} \cap Y_\Gamma))}$$

whose period integral is given by the log-divergent parametric Feynman integral above?

If so

is this motive mixed Tate? Are there specific combinatorial requirements on Γ that makes it mixed Tate?

Major result due to Bloch–Esnault–Kreimer: For the wheel-with-n-spokes graph with $n \geq 3$, the Hodge–Tate realization is $\mathbb{Q}(2n-3)$. From earlier work by Broadhurst–Kreimer, the corresponding integrals, after divergence issues have been resolved, evaluates to $c_n \zeta(2n-3)$ where $c_n \in \mathbb{Q}$.

Main results so far

1. Broadhurst–Kreimer (90's) with the help of the newly discovered Hopf algebra structure of Feynman graphs evaluate large number of graphs with many loops and discover zeta values, multiple zeta values and values of multipolylog.

2. Goncharov–Manin in their study of $\overline{M_{0,n}}$ and Deligne suggest that MZVs are periods of mixed Tate motives. Goncharov–Manin show that MZVs are periods of $\overline{M_{0,n}}$.

3. Kontsevich conjectures (on based of a computer experiment) that $\#\mathbb{F}_q$ points of X_Γ (up to a certain number of loops) is a polynomial in q suggesting all graph hypersurfaces are mixed Tate.

4. Belkale–Brosnan disproves this conjecture and instead establishes that

   $$\boxed{\text{graph hypersurfaces generate the full ring of } K_0(\mathbf{Var}_k).}$$

   Put another way, the "generic" graph might be not mixed Tate.

5. Flurry of activities after that. Main players: Aluffi, Bloch, F. Brown, Esnault, Kreimer, Marcolli and their students and post-docs. Nice summary of the activities till 2009– M. Marcolli, *Feynman motives*, World Scientific, 2010.

6. Francis Brown very recently establishes something big that was "in the air" for sometimes: MZVs are periods of mixed Tate motives over Spec $\mathbb{Z}$.

7. In parallel there is a theory of Connes–Marcolli relating quantum field theory and renormalization to Tannakian categories and differential Galois theory. See A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*, AMS/Hindustan Books, 2008. At present it is not clear how these two approaches might relate.

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Generating supermanifolds from realistic Feynman graphs

By a complex supermanifold [Manin] one understands a datum

$$\boxed{\mathcal{X} = (X, \mathcal{A})}$$

with the following properties:

1. $\mathcal{A}$ is a sheaf of supercommutative rings on X,
2. $(X, \mathcal{O}_X)$ is a complex quasiprojective algebraic variety, where $\mathcal{O}_X = \mathcal{A}/\mathcal{N}$, with $\mathcal{N}$ the ideal of nilpotents in $\mathcal{A}$ and,
3. the quotient $\mathcal{E} = \mathcal{N}/\mathcal{N}^2$ is locally free over $\mathcal{O}_X$ and $\mathcal{A}$ is locally isomorphic to the exterior algebra $\Lambda^\bullet_{\mathcal{O}_X}(\mathcal{E})$, where the grading is the $\mathbb{Z}_2$-grading by odd/even degrees.

The supermanifold is split if $\mathcal{A} \simeq \Lambda^\bullet_{\mathcal{O}_X}(\mathcal{E})$ is global.

A canonical example is the projective superspace $\mathbb{P}^{n|m}$ with $X = \mathbb{P}^n$ and

$$\mathcal{A} = \Lambda^\bullet(\mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{O}(-1))$$

with exterior powers $\Lambda^\bullet$ given by odd/even degrees.

These objects naturally in the context of mirror symmetry!

Generalization of log-divergent integrals

Theorem (M.Marcolli – AR). Suppose given a graph with n edges of which f fermionic and $b = n - f$ bosonic. Assume there exists a choice of basis of $H_1(\Gamma, \mathbb{Z})$ satisfying

$$n - \frac{f}{2} - \frac{D}{2}(\ell_f - \ell_b) = 0.$$

Then the following identity holds:

$$\boxed{\int \frac{\not{q}_1 \cdots \not{q}_f}{q_1 \cdots q_n} d^D s_1 \cdots d^D s_{\ell_b} d^D \sigma_1 \cdots d^D \sigma_{\ell_f} = \int_{\Sigma_n} \frac{\Lambda(t)}{\mathrm{Ber}\mathcal{M}(t)^{D/2}} dt_1 \cdots dt_n.}$$

Here:

- $q(p) = p^2 - m^2, \quad \not{q}(p) = i(\not{p} + m), \quad \not{p} = p^\mu \gamma_\mu$
- $\Lambda(t)$: some term given by powers of $M_f(t)$
- $\mathcal{M}(t) = \begin{pmatrix} M_b(t) & \frac{1}{2} M_{fb}(t) \\ \frac{1}{2} M_{bf}(t) & M_f(t) \end{pmatrix}, \quad \mathrm{Ber}\mathcal{M} = \frac{\det(M_b - \frac{1}{4} M_{fb} M_f^{-1} M_{bf})}{\det M_f}$

Note that the matrices $M_f(t)$, $M_b(t)$, $M_{bf}(t)$ and $M_{fb}(t)$ where all obtained as with the case of scalar theory by looking at incidence matrices though this time taking care of two different kinds of variables leading to two different kinds of loop variables $\{s_1, \ldots, s_{\ell_b}\}$ (ordinary) and $\{\sigma_1, \ldots, \sigma_{\ell_f}\}$ (grassmannian). In other words- $s_- \in \mathbb{A}^{D|0}$ while $\sigma_- \in \mathbb{A}^{0|D}$.

Graph supermanifolds

Divergences occur when Σ_n intersects with the subvariety of $\mathbb{P}^{n-1}$ defined by

$$\frac{\mathrm{Ber}\mathcal{M}(t)^{D/2}}{\Lambda(t)} = 0 \qquad (*)$$

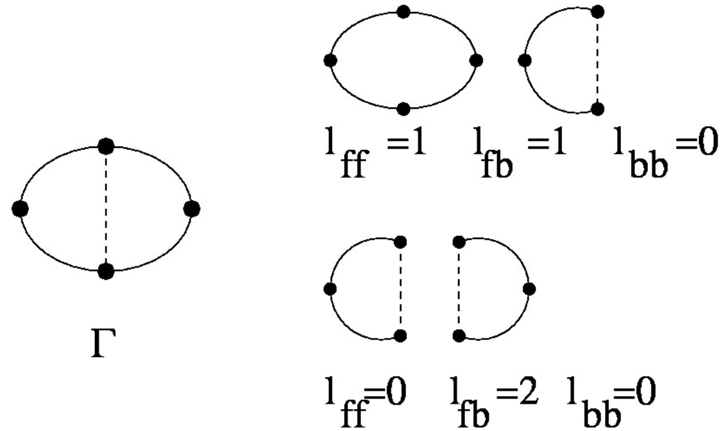
A lemma tells us that the zeros of $(*)$ define a divisor in $\mathbb{P}^{n-1|2f}$ of dim. $(n-2|2f)$. The support of this divisor is the same as that of the principal divisor defined by $\mathrm{Ber}\mathcal{M}(t)$.

Let Γ be a graph with bosonic and fermionic edges and B an admissible basis for $H_1(\Gamma, \mathbb{Z})$. Define

$$\boxed{\mathcal{X}_{(\Gamma,B)} \subset \mathbb{P}^{n-1|2f}}$$

to be the locus of zeros and poles of $\mathrm{Ber}\mathcal{M}(t)$. $\mathcal{X}_{(\Gamma,B)}$ is called the graph supermanifold.

The issue of admissible basis for a given supergraph (D = 6)



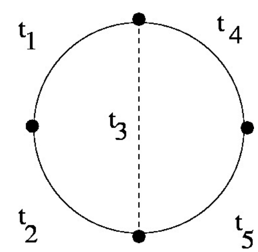
Remembering our generalization of the log-divergence condition,

$$
\begin{aligned}
& n-\frac{f}{2}-\frac{D}{2}(\ell_f-\ell_b) \\
\stackrel{\text{def}}{=} \quad & n-\frac{f}{2}-\frac{D}{2}((\ell_{ff}+\ell_{fb})-(\ell_{bb}+\ell_{fb}))=0
\end{aligned}
$$

we see that for the graph on the left, only the basis at the top satisfies it

$$5-\frac{4}{2}-\frac{6}{2}((1+1)-(0+1))=0.$$

Example ($D = 6$)



Here

$$M_b(t)=t_1+t_2+t_3, \quad M_{bf}(t)=(t_1+t_2,t_1+t_2+t_3), \quad M_f(t)=\begin{pmatrix} 0 & t_1+t_2 \\ -(t_1+t_2) & 0 \end{pmatrix}$$

Therefore

$$M_{bf}(t)M_f(t)^{-1}M_{fb}(t) = -(t_1+t_2+t_3)+t_1+t_2+t_3 \equiv 0$$

and

$$\mathrm{Ber}\mathcal{M}(t) = \frac{\det(M_b(t) - \frac{1}{4}M_{bf}(t)M_f(t)^{-1}M_{fb}(t))}{\det M_f(t)} = \frac{\det M_b(t)}{\det M_f(t)} = \frac{t_1+t_2+t_3}{(t_1+t_2)^2}.$$

So, $\mathcal{X}_{(\Gamma,B)} \subset \mathbb{P}^{4|8}$ is the union of $t_1+t_2+t_3 = 0$ and $t_1+t_2 = 0$ (counted with multiplicity two) in $\mathbb{P}^4$ with restriction of the sheaf from $\mathbb{P}^{4|8}$.

Grothendieck ring, motives and supermanifolds

Let $\mathbf{SVar}_{\mathbb{C}}$ be the category of complex supermanifolds. Let $K_0(\mathbf{SVar}_{\mathbb{C}})$ denote the free abelian group generated by isom. classes of objects $\mathcal{X} \in \mathbf{SVar}_{\mathbb{C}}$ subject to the following:

> Let $F : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed embedding of supermanifolds. Then
>
> $$[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} \setminus \mathcal{Y}],$$
>
> where $\mathcal{X} \setminus \mathcal{Y}$ is the supermanifold
>
> $$\mathcal{X} \setminus \mathcal{Y} = (X \setminus Y, \mathcal{A}_{X|X\setminus Y}).$$

Here $\mathcal{A}$ is a sheaf of supercommutative rings on X.

The following proposition [M.Marcolli–AR] relates the two Grothendieck rings:

$$\boxed{K_0(\mathbf{SVar}_{\mathbb{C}}) = K_0(\mathbf{Var}_{\mathbb{C}})[T]}$$

where $T = [\mathbb{A}^{0|1}]$ class of the affine superspace of dim. $(0,1)$. In analogy to the description of $K_0(\mathbf{Var}_k)$ we also have

$$K_0(\mathcal{SV}_{\mathbb{C}})/I \simeq \mathbb{Z}[SSB]$$

where I is the ideal gen. by $[\mathbb{A}^{0|1}]$, $[\mathbb{A}^{1|0}]$.
NB: There are two different kinds of Lefschetz motives in supergeometry $\mathbb{L}_f = [\mathbb{A}^{0|1}]$, $\mathbb{L}_b = [\mathbb{A}^{1|0}]$.

A direct corollary of our theorem is a universality result analogous to the Belkale–Brosnan–

Let $\mathcal{R}$ be the subring of the Grothendieck ring $K_0(\mathbf{SVar}_{\mathbb{C}})$ spanned by $[\mathcal{X}_{(\Gamma,B)}]$ for $\mathcal{X}_{(\Gamma,B)}$ given by the zeros and poles of the Berezinian $\mathrm{Ber}\mathcal{M}(t)$ with B a chosen basis for $H_1(\Gamma,\mathbb{Z})$. Then

$$\boxed{\mathcal{R} = K_0(\mathbf{Var}_{\mathbb{C}})[T^2] \subset K_0(\mathbf{SVar}_{\mathbb{C}})}$$

where $T = [\mathbb{A}^{0|1}]$.

Lie and Hopf algebras of Feynman graphs

The Connes–Kreimer Hopf algebra:

$$\Delta(\Gamma) = \underbrace{1 \otimes \Gamma + \Gamma \otimes 1}_{\text{definition of primitive graphs}} + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

Application of the Milnor-Moore theorem: Dual to the CK Hopf algebra of deletion and contraction, the Lie algebra $\mathcal{L}_{CK}$ has the Lie bracket

$$[\Gamma, \Gamma'] = \left(\sum_{\text{all vertices of } \Gamma} \Gamma \leftarrow \Gamma' \right) - \left(\sum_{\text{all vertices of } \Gamma'} \Gamma' \leftarrow \Gamma \right)$$

Kremnizer-Szczesny:

- $K_0(\mathbf{FGph}) \simeq \mathbb{Z}[\mathcal{P}]$, $\mathcal{P}$ primitive graphs.

- $\mathbf{FGph}$ finitary abelian category and $\mathcal{L}_{CK}$ Ringel-Hall algebra associated to $\mathbf{FGph}$.

Problem: Lift CK insertion to the level of graph polynomials. Bloch–Esnault–Kreimer gives us the following formula

$$\Psi_\Gamma = \Psi_\gamma \Psi_{\Gamma/\gamma} + f(t_1, \ldots, t_m)$$

where f is of $\deg(f) < h_1(\gamma)$ and $m = \#E(\Gamma/\gamma)$.

In joint work with Christoph Bergbauer, we found an explicit formula expression of f (dually) in case of $\Gamma \leftarrow \gamma$, i.e., relate Ψ_Γ, Ψ_γ and $\Psi_{\Gamma \leftarrow \gamma}$. Our explicit formula is

$$\boxed{\Psi_{\Gamma \leftarrow \gamma} = \Psi_\gamma \Psi_\Gamma + \sum_{0 \neq P \leq P_v} \Psi_{\gamma^P} \tilde{\Psi}_{\Gamma^P}}$$

The second term involves rather careful combinatorics on the complete graph in which we view Γ, γ and $\Gamma \leftarrow \gamma$ to be “embedded in”. It has given us, as corollaries nice results about the singular locus of Γ. Further computation of characteristic classes using this formula is underway!

Shameless self-promotion!

1. [with C. Bergbauer], *Insertion of graphs and singularities of graph hypersurfaces*, preprint.

2. *Motives: An introductory survey for physicists*, Contemp. Math. **539**, pp. 377–415 (2011)

3. [with M. Marcolli] *Supermanifolds from Feynman graphs*, J.Phys A: Math. Theor. **41** (2008), 315402